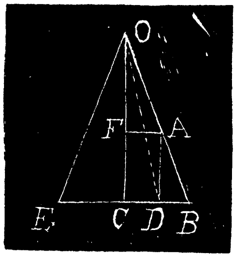
52. Proposed by I. J. WIREBACK, M. D., St. Petersburg, Penn.

What is the volume of a segment of a right cone, whose diameter is 6 inches and perpendicular 9 inches? The section being parallel with the perpendicular of the cone and includes $\frac14$ of its circumference at the base.

I. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let OBE be section of cone perpendicular to section cutting off segment ADB. By considering projection of hyperbolic section AD on parallel plane through the axis, it is seen that the asymptotes are intersections of latter plane with conical surface. Accordingly, if $OF=a$, $FA=b$, equation to hyperbola is



$$a^2y^2-b^2x^2=-a^2b^2.$$

Now $FA=CD=\frac12$ side of square inscribed in circle $EB=\frac32\sqrt2=b$. $OF:FA=OC:CB$, or $OF:\frac32\sqrt2=9:3$; $OF=\frac92\sqrt2=a$. Substituting in formula for area of hyperbola,

$$\text{area } AD=(b/a)x\sqrt{x^2-a^2}-ab\log_e\left(\frac{x+\sqrt{x^2-a^2}}{a}\right),$$

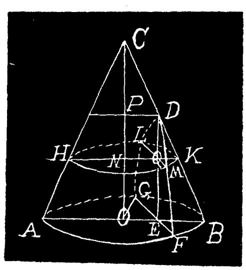
$$=\tfrac13\times9\sqrt{81-\tfrac{81}{2}}-27\log_e\left(\frac{9+\sqrt{81-\frac{81}{2}}}{\frac92\sqrt2}\right),$$

$$=\tfrac{27}{2}\sqrt2-\tfrac{27}{2}\log_e(\sqrt2+1).$$

Volume of conical segment $OAD=\frac13CD\times$ area AD, $=\frac{27}{2}-\frac{27}{4}\sqrt2\log_e(\sqrt2+1)$.
Area of circular segment $BD=9\pi/4-\frac92$, $=\frac94(\pi-2)$.
Volume of conical segment $OBD=\frac13OC\times$ area DB, $=\frac{27}{4}(\pi-2)$.
Volume $ABD=$ volume $OBD-$ volume $OAD=\frac{27}{4}(\pi-2)-\frac{27}{2}+\frac{27}{4}\sqrt2\log_e(\sqrt2+1)$
$=\frac{27}{4}\pi-27+\frac{27}{4}\sqrt2\log_e(\sqrt2+1)$, $=2.619+$ cubic inches.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let $AFB-C$ be the cone, $GLDMF$ the section made by the plane cutting off the given segment. Let $AB=6=2R$, $OC=9=h$, $OE=c$. Since $\angle GOF=\pi/2$, $GF=R\sqrt2$.



$\therefore OE=\sqrt{OG^2-GE^2}=\sqrt{R^2-\frac12R^2}=\frac12R\sqrt2$
$=\frac12(3\sqrt2)=c$.

Let $CN=x$, then $CO:OB=CN:NK$.
or $h:R=x:NK$. $\therefore NK=Rx/h=NL$.

Similarly $CP=CO-DF=h-\dfrac{h(R-c)}{R}=\dfrac{hc}{R}$.

Area of segment $LMK=\dfrac{R^2x^2}{h^2}\cos^{-1}\left(\dfrac{ch}{Rx}\right)-\dfrac{c}{h}\sqrt{R^2x^2-h^2c^2}$.

$$\therefore\ V=\int_{ch/R}^{h}\left\{\frac{R^2x^2}{h^2}\cos^{-1}\left(\frac{ch}{Rx}\right)-\frac{c}{h}\sqrt{R^2x^2-h^2c^2}\right\}dx,$$

$$=\tfrac{1}{3}h\left\{R^2\cos^{-1}\left(\frac{c}{R}\right)-2c\sqrt{R^3-c^2}+\frac{c^3}{R}\log\left(\frac{R+\sqrt{R^2-c^2}}{c}\right)\right\},$$

$$=3\{9\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)-9+\frac{9\sqrt{2}}{4}\log(\sqrt{2}+1)\},\ =2.619\text{ cubic inches.}$$

III. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

I Solution. Designating the radius of the base by r, the altitude by h, and choosing the center of the base for the origin of orthogonal coördinates, CO for the axis of z, the radius OB for the axis of x and a radius parallel to the section FDG for that of y, we find the equation of the cone to be

$$z=(h/r)(r-\sqrt{x^2+y^2},$$

and the volume V of $COFGD$

$$=\int_{-\sqrt{r^2-x^2}}^{-\sqrt{r^2+x^2}}zdy=\frac{h}{r}\int_{o}^{x}dx\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}}(r-\sqrt{x^2+y^2})dy,$$

$$=\frac{h}{r}\int_{o}^{x}\left(r\sqrt{r^2-x^2}-x^2\log\frac{r+\sqrt{r^2-x^2}}{x}\right)dx,$$

$$=\tfrac{2}{3}hx\sqrt{r^2-x^2}+\tfrac{1}{3}r^2h\sin^{-1}(x/r)-\tfrac{1}{3}\frac{hx^3}{r}\log\frac{r+\sqrt{r^2-x^2}}{x}.$$

Substituting $x=(r/2)\sqrt{2}$, we have for the volume $COFGD$ the expression $\frac{r^2h}{12}[4+\pi-\sqrt{2}.\log(\sqrt{2}+1)]$, and for that of $B-FGD$ $\frac{r^2h}{12}[\pi-4+\sqrt{2}.\log(\sqrt{2}+1)]$

II Solution. Let HK be a circle parallel to AB cutting the hyperbola FDK in the points L and M, and let the diameter HK cut the axis DE at Q. Put $OE=b$, $OF=r$, $CO=h$, $DQ=x$, $LQ=y$. We find from the geometry of the figure $y^2=(2br/x)x+(r^2/h^2)x^2$ as the equation of the hyperbola FDK.

$\therefore$ Area of $FDK=2\int dx\sqrt{\frac{2br}{h}x+\frac{r^2}{h^2}x^2}$ between the limits O,

and $DE=\frac{(r-b)h}{r}$. Integrating we find for this area the expression,

$$\frac{h}{b}\left(r\sqrt{r^2-b^2}-b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right).$$

$$\therefore\ \text{Volume of } COFGD=\frac{h}{b}\int_{o}^{b}\left(r\sqrt{r^2-b^2}=b^2\log\frac{r+\sqrt{r^2-b^2}}{b}\right)db$$

$$=\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r}-\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{b}\Big),$$

and volume of $BFGD=\tfrac{1}{3}r^2h\cos^{-1}\dfrac{b}{r}-\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}\dfrac{hb^3}{r}\log\dfrac{r+\sqrt{r^2-b^2}}{b}$.

III Solution. Let HK be a circle parallel to AB, and N its centre. Through N draw a diameter parallel to the hyperbolic section FGD. Put $CN=x$, $OE=b$, $BO=r$, $CO=h$, then the area of the circular segment lying between the diameter through N and the parallel chord LM

$$=\frac{r^2x^2}{h^2}\sin^{-1}\frac{bh}{rx}+\frac{br}{h}\sqrt{x^2-\frac{b^2h^2}{r^2}}.$$

$\therefore$ Volume of conical section $COFGD$

$$=\frac{r}{h}\left\{\frac{r}{h}\int x^2\sin^{-1}\frac{bh}{rx}+b\int dx\sqrt{x^2-\frac{b^2h^2}{r^2}}\right\},$$

the integrals to be taken between $h-DE=bh/r$ and h. Thus we find for this volume the expression

$$\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}r^2h\sin^{-1}\frac{b}{r}-\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r};$$

and for the volume of the conical section $DBFG$,

$$\tfrac{1}{3}r^2h\cos^{-1}\frac{b}{r}-\tfrac{2}{3}hb\sqrt{r^2-b^2}+\tfrac{1}{3}\frac{hb^3}{r}\log\frac{r+\sqrt{r^2-b^2}}{r}.$$

HISTORICAL NOTE. The famous astronomer KEPLER tried hard to find the volume of such conical sections as the above, but all his efforts proved futile.

Also solved by *GEORGE LILLEY, Ph. D., LL. D.*, and *CHARLES C. CROSS*. Dr. Lilley obtained a numerical result of 6.771 cubic inches, and Professor Cross obtained 2.256979 cubic inches.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

87. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

A and B set out from the same place, and in the same direction. A travels uniformly 18 miles per day, and after 9 days turns and goes back as far as B has traveled during those 9 days; he then turns again, and, pursuing his journey, overtakes B 22½ days after the time they first set out. It is required to find the rate at which B uniformly traveled. [From *Greenleaf's Arithmetic*.]

88. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the principal of a note given March 19, 1891, bearing interest at 6%. Payments: September 1, 1892, \$243.50; January 19, 1893, \$6.90; April 13, 1894, \$19.10; September 19, 1894, \$110.90. Amount due February 22, 1897, \$229.10.